

# The asymptotic behaviour of the exact and approximative $\nu = 1/2$ Chern-Simons Green's functions

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**Abstract.** We consider the asymptotic behaviour of the Chern-Simons Green's function of the  $\nu = 1/\tilde{\phi}$  system for an infinite area in position-time representation. We calculate explicitly the asymptotic form of the Green's function of the interaction free Chern-Simons system for small times. The calculated Green's function vanishes exponentially with the logarithm of the area. Furthermore, we discuss the form of the divergence for all  $\tau$  and also for the Coulomb interacting Chern-Simons system. We compare the asymptotics of the exact Chern-Simons Green's function with the asymptotics of the Green's function in the Hartree-Fock as well as the random-phase approximation (RPA). The asymptotics of the Hartree-Fock Green's function correspondence well with the exact Green's function. In the case of the RPA Green's function we do not get the correct asymptotics. At last, we calculate the self consistent Hartree-Fock Green's function.

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## 1 Introduction

The combination of an electronic interaction and a strong magnetic field in a two-dimensional electron system yields a rich variety of phases. These are best classified by the filling factor  $\nu$ , which is the electron density divided by the density of a completely filled Landau level. In this paper, we consider systems with filling fraction  $\nu = 1/\tilde{\phi}$  where  $\tilde{\phi}$  is an even positive number. These systems are most suitably described by the Chern-Simons theory. Since the discovery of the fractional quantum Hall effect by Tsui *et al.* [1] there were many attempts to explain this experimental observation. The contemporary theoretical picture of this effect is mainly based on a work of Jain [2]. In his theory he mapped the wave functions of the integer quantum Hall effect to wave functions of the fractional quantum Hall effect. In the case of filling fraction  $\nu = 1/2$  every electron gets two magnetic flux quanta through this mapping. With the help of this transformation new quasi-particles (composite fermions) are obtained which do not see any magnetic field in first approximation (mean field). A field theoretical language for this scenario was first established by Halperin, Lee, Read (HLR) [3], as well as Kalmeyer and Zhang [4] for the  $\nu = 1/2$  system. The interpretation of many experiments supports this composite fermion picture. We mention transport experiments with quantum (anti-) dots [5], and focusing experiments [6]

here. An overview of further experiments can be found in [7].

HLR studied many physical quantities within the random-phase approximation (RPA). Most prominent among these is the effective mass of the composite fermions which they found to diverge at the Fermi surface [3,8]. This is based on the interaction of the composite fermions *via* transversal gauge interactions. Later on, Shankar and Murthy [9] proposed a new theory of the  $\nu = 1/2$  system. Based upon a transformation of the Chern-Simons Hamiltonian one achieves a separation of the magneto-plasmon oscillators from the total interaction of the system. After restricting the number of the magneto-plasmon oscillators to the number of electrons Shankar and Murthy got a finite quasi-particle mass which scales with the inverse of the strength of the Coulomb repulsion. In their derivation they calculated a smaller number of self energy Feynman diagrams than in the RPA. Recently Stern *et al.* [10] calculated the self energy of the theory of Shankar and Murthy in RPA finding the same divergence of the effective mass as HLR. Besides the theories of HLR and Shankar and Murthy there are other alternative formulations of the Chern-Simons theory which looks similar to the Chern-Simons theory of Shankar and Murthy [11].

In this paper, we consider the asymptotic behaviour of the  $\nu = 1/\tilde{\phi}$  Chern-Simons Green's function for an infinite area  $A$  non-perturbationally. It is well known that the Hartree-Fock approximation [12] as well as the RPA [3]

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of the Chern-Simons Green's function results in a  $\log(A)$  singularity in the momentum-frequency representation  $(\mathbf{q}, \omega)$  of the self energy. This singularity is caused by the interaction of the composite fermions through a longitudinal gauge field (in contrast to the effective mass). In almost all calculations of the effective mass of the composite fermions this singularity is not taken into account although one can easily show that this  $\log(A)$  singularity enforces the effective mass to be finite (the bare mass). Furthermore, this singularity of the self energy is not only given on the Fermi-surface but for all momenta (apart from the Fermi-surface). The neglect of the  $\log(A)$  singularity in the calculation of the effective mass is justified by the physical argument that it must scale with the inverse of the Coulomb interaction [3]. By neglecting this singularity one gets the correct scaling. This was the reason that till now almost all authors disregard the  $\log(A)$  singularity and try to get rather a better physical insight into the effective mass singularity caused by the transverse gauge interaction ([13] and references therein). From the point of view that the Chern-Simons theory is a many-body theory for which physical quantities are calculated by known perturbative methods (although no small parameter is present) this procedure is not satisfactory. On the way to integrate this singularity in a Chern-Simons perturbation theory or to formulate a Chern-Simons theory without this Green's function divergence, we will investigate the  $\log(A)$  singularity in this paper. The aim of our investigation is to get rather exact statements about this singularity and the relation of perturbative calculated Green's functions to the exact Green's function.

HLR [3,14] gave in their paper a semi-classical reason for the  $\log(A)$  singularity by showing that the Chern-Simons transformation effectively gives a velocity boost to every electron. This velocity boost results in a one particle energy which diverges proportional to  $\log(A)$ . Unfortunately, the derivation of HLR is based on semi-classical approximations. To our knowledge there are no publications which show without approximations that the  $\log(A)$  singularity is really existent in the Green's function or even how it looks like. In this paper, we will calculate the asymptotics of the Chern-Simons Green's function in the position-time  $(\mathbf{r}, \tau)$  representation. The asymptotics will be computed concretely for the interaction free  $\nu = 1/\tilde{\phi}$  Chern-Simons system for all  $\mathbf{r}$  and small  $\tau$ . We will show exactly that the Green's function vanishes for  $A \rightarrow \infty$ . Furthermore, we will discuss the form of the divergence for all  $\tau$  and also for the Coulomb interacting Chern-Simons system. It is clear that perturbational calculations of physical quantities such as the energy should start with Green's and vertex functions which are in rather good agreement with the exact functions. To obtain a good approximation of the Chern-Simons Green's function, we will calculate it in two different approximations. First, we will calculate the Hartree-Fock Green's function. We will show that the asymptotic behaviour of this Green's function is in good agreement with the asymptotic behaviour of the exact Green's function. Then we will calculate the Chern-Simons Green's function in RPA. We will show that this

Green's function is finite for  $A \rightarrow \infty$  (modulo logarithmic singularities), which is not in agreement with the exact Green's function. On the way to formulate a perturbation theory around the Hartree-Fock mean field, we will examine at last the self consistent Hartree-Fock Green's function.

In the following derivation of the Chern-Simons Green's function, we will keep the formulas as general as possible. This will be done to simplify the extension of the concrete calculation of the asymptotics of the Green's function  $G(\mathbf{r}, \tau)$  also for large  $\tau$  in a later publication. In this publication, we restrict our calculation of the Green's function to the range of small  $\tau$  because this  $\tau$  range is most relevant for a calculation of physical quantities (the Green's function decreases by physical arguments for larger  $\tau$ ). Furthermore, we restrict our calculations to the case temperature  $T = 0$ .

The paper is organized as follows:

In Section 2, we will calculate the asymptotic behaviour of the Chern-Simons Green's function  $G(\mathbf{r}, \tau)$ . In Section 3, we will calculate the self consistent Chern-Simons Green's function in Hartree-Fock approximation as well as in RPA and compare these approximations with the exact Chern-Simons Green's function calculated in Section 2.

## 2 The asymptotic behaviour of the Chern-Simons Green's function

In this paper we consider interacting spin polarized electrons moving in two dimensions in a strong magnetic field  $B$  directed in the negative  $z$ -direction of the system. The electronic density of the system is chosen such that the lowest Landau level of a non-interacting system is filled to a fraction  $\nu = 1/\tilde{\phi}$  where  $\tilde{\phi}$  is an even number. We are mainly interested in  $\tilde{\phi} = 2$ . The composite fermion annihilation operator  $\Psi(\mathbf{r})$  is defined from the electronic annihilation operator  $\Psi_e(\mathbf{r})$  through the Chern-Simons transformation [15] ( $\alpha(\mathbf{r})$  is the angle between the  $x$ -axes and the position  $\mathbf{r}$ )

$$\Psi(\mathbf{r}) = e^{-i\phi \int d^2r' \alpha(\mathbf{r}-\mathbf{r}') \Psi^\dagger(\mathbf{r}') \Psi(\mathbf{r}')} \Psi_e(\mathbf{r}). \quad (1)$$

The Hamiltonian of the composite fermions is then given by:

$$H_{CS} = \int d^2r \left[ \frac{1}{2m} |(-i\nabla + \mathbf{A} + \mathbf{a}_{CS})\Psi(\mathbf{r})|^2 + \frac{1}{2} \int d^2r' \left\{ (|\Psi(\mathbf{r})|^2 - \rho_B) V^{ee}(|\mathbf{r}-\mathbf{r}'|) (|\Psi(\mathbf{r}')|^2 - \rho_B) \right\} \right]. \quad (2)$$

The Chern-Simons vector potential  $\mathbf{a}_{CS}$  is defined by  $\mathbf{a}_{CS}(\mathbf{r}) = \tilde{\phi} \int d^2r' \mathbf{f}(\mathbf{r}-\mathbf{r}') \Psi^\dagger(\mathbf{r}') \Psi(\mathbf{r}')$ .  $m$  is the mass of an electron.  $V^{ee}(r) = e^2/r$  is the Coulomb interaction where  $e^2 = q_e^2/\epsilon$ .  $q_e$  is the charge of the electrons and  $\epsilon$  is the dielectric constant of the background field  $\rho_B$ .  $\mathbf{A}(\mathbf{r})$  is the vector potential  $\mathbf{A} = 1/2 \mathbf{B} \times \mathbf{r}$  and  $\mathbf{B}$  is

a homogeneous magnetic field in the negative  $z$ -direction  $\mathbf{B} = -B\mathbf{e}_z$ , where  $\mathbf{e}_z$  is the unit vector in  $z$ -direction. We suppose throughout this paper that  $B$  is a positive number. The function  $\mathbf{f}(\mathbf{r})$  is given by  $\mathbf{f}(\mathbf{r}) = \mathbf{e}_z \times \mathbf{r}/r^2$ . We used the convention  $\hbar = 1$  and  $c = 1$  in the above formula (2). Furthermore, we set  $q_e = 1$  for the coupling of the magnetic potential to the electrons. The Green's function of the Chern-Simons theory is defined as

$$G(\mathbf{r}, t; \mathbf{r}', t') = \langle T[\Psi(\mathbf{r}, t), \Psi^\dagger(\mathbf{r}', t')] \rangle_{\text{CS}}. \quad (3)$$

$\langle \cdot \rangle_{\text{CS}}$  is the average over the Gibb's operator of  $H_{\text{CS}}$ .  $T$  is the time ordering operator. Applying the inverse of the Chern-Simons transformation (1) to the operator  $H_{\text{CS}}$ , we get the electronic Hamiltonian  $H_e$ .  $H_e$  is given by  $H_{\text{CS}}$  (2) via the substitutions  $a_{\text{CS}} = 0$ ,  $\Psi \rightarrow \Psi_e$  and  $\Psi^\dagger \rightarrow \Psi_e^\dagger$ .

As mentioned in the introduction, we will at first calculate the  $N$  particle Green's function for  $V_{ee} = 0$ . Under this restriction the ground state of  $H_e$  is degenerate. The ground state wave functions are given by

$$u_{0,\mathbf{p}} = S[u_{p_1}, u_{p_2}, \dots, u_{p_N}]. \quad (4)$$

$S$  is the Slater determinant of the one particle wave functions  $u_{p_1}, \dots, u_{p_N}$ .  $u_p$  are the one particle wave functions in the lowest Landau level in the symmetric gauge

$$u_p(r, \phi) = \sqrt{\frac{1}{2^{p+1} p! \pi}} e^{ip\phi} r^p e^{-\frac{1}{4}r^2}. \quad (5)$$

In equation (4), we have  $p \in 0 \dots \tilde{\phi}N$  (for finite  $A$ ). Further, we use the simplifications  $B = 1$ ,  $m = 1$ . This will be done throughout this section in the auxiliary equations. For equations which have the character of a result, we will insert the  $B$  and  $m$  dependencies explicitly. In the following, we have to calculate the expectation values of certain differential-position operators with respect to the ground state wave functions. Since the norm of  $u_p$  is one we obtain for the expectation value of every differential operator of first order with respect to  $u_p$  a multiplicative factor  $\sqrt{p}$  in the result. We get the same multiplicative factor  $\sqrt{p}$  for every position operator in the differential-position operator. Thus, we can easily read off the leading  $p$  dependence of the average of a differential-position operator with respect to a lowest Landau level wave function  $u_p$ . Similarly, one can calculate the result of the average of a differential-position operator over the many particle ground state wave function  $u_{0,\mathbf{p}}$ .

In order to calculate the asymptotic behaviour of the Green's function (3), we have organized this section as follows: In Section 2.1, we will calculate the asymptotics of the Green's function  $G(\mathbf{r}, \tau)$  for  $\mathbf{r} = 0$  and  $\tau > 0$ . In Section 2.2 we will extend the calculation to  $\mathbf{r} \neq 0$ . In Section 2.3, we take the average of the asymptotic expressions of Sections 2.1, 2.2 with respect to the ground states of filling fraction  $\nu = 1/\tilde{\phi}$ . The asymptotics of the Green's function  $G(\mathbf{r}, \tau)$  for  $\tau < 0$  will be calculated in Section 2.4. In Section 2.5, we consider the Chern-Simons Green's function by taking into account the Coulomb interaction between the electrons.

## 2.1 The calculation of the asymptotic behaviour of the Chern-Simons Green's function $G(0, \tau)_{\tau > 0}$

In this subsection, we calculate the asymptotics of the Chern-Simons Green's function  $G(0, \tau)_{\tau > 0}$  for  $\tau > 0$ . In the following, we apply the inverse of the Chern-Simons transformation to the expression (3). The result will be represented in the one particle basis. We get

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{e^{((t-t')-\beta)(\frac{1}{2}-\mu)N}}{Z} \times \sum_{u_{0,\mathbf{k}} \in \nu_{1/\tilde{\phi}}} \left| A \left[ u_{0,\mathbf{k}}(\mathbf{r}_1, \dots, \mathbf{r}_N) \delta_{\mathbf{r}}(\mathbf{r}_{N+1}) \right] \right| \exp \left[ -i\tilde{\phi} \left( \sum_{i=1}^{N+1} \alpha(\mathbf{r}_i - \mathbf{r}) - \alpha(\mathbf{r}_i - \mathbf{r}') \right) \right] \left| A \left[ \exp \left[ - (t - t') (H_{N+1} + \overline{H}_{N+1}) \right] u_{0,\mathbf{k}}(\mathbf{r}_1, \dots, \mathbf{r}_N) \delta_{\mathbf{r}'}(\mathbf{r}_{N+1}) \right] \right|. \quad (6)$$

The Hamilton operators  $H_{N+1}$ ,  $\overline{H}_{N+1}$  are given by

$$H_{N+1} = \frac{1}{2} \left[ -i\nabla_{N+1} + \mathbf{A}(\mathbf{r}_{N+1}) + \tilde{\phi} \sum_{i=1}^N \mathbf{f}(\mathbf{r}_{N+1} - \mathbf{r}_i) \right]^2, \quad \overline{H}_{N+1} = \sum_{i=1}^N \frac{1}{2} \left[ -i\nabla_i + \mathbf{A}(\mathbf{r}_i) + \tilde{\phi} \mathbf{f}(\mathbf{r}_i - \mathbf{r}_{N+1}) \right]^2. \quad (7)$$

Here  $\beta = 1/(k_B T)$  where  $k_B$  is the Boltzmann constant.  $Z$  is the partition function of the Hamiltonian  $H_e$ .  $A$  is the antisymmetrization operator.  $\nu_{1/\tilde{\phi}}$  are the ground states of the  $\nu = 1/\tilde{\phi}$  system. We split  $H_{N+1} + \overline{H}_{N+1}$  in its components  $H_{N+1} + \overline{H}_{N+1} = H_0 + H_1 + H_2 + H_3 + H_{4,1} + H_{4,2}$  by

$$H_{0,1} = \sum_{i=1}^N \frac{1}{2} \left[ -i\nabla_i + \mathbf{A}(\mathbf{r}_i) \right]^2 - \mu, \quad (8)$$

$$H_{0,2} = \frac{1}{2} \left[ -i\nabla_{N+1} + \mathbf{A}(\mathbf{r}_{N+1}) \right]^2 - \mu, \quad (9)$$

$$H_1 = \sum_{i=1}^N \tilde{\phi}^2 \mathbf{f}(\mathbf{r}_i - \mathbf{r}_{N+1})^2, \quad (10)$$

$$H_2 = \frac{1}{2} \sum_{i \neq j=1}^N \tilde{\phi}^2 \mathbf{f}(\mathbf{r}_i - \mathbf{r}_{N+1}) \mathbf{f}(\mathbf{r}_j - \mathbf{r}_{N+1}), \quad (11)$$

$$H_3 = \sum_{i=1}^N \mathbf{A}(\mathbf{r}_i) \tilde{\phi} \mathbf{f}(\mathbf{r}_i - \mathbf{r}_{N+1}) - \mathbf{A}(\mathbf{r}_{N+1}) \tilde{\phi} \mathbf{f}(\mathbf{r}_i - \mathbf{r}_{N+1}) = -N \frac{\tilde{\phi}}{2}, \quad (12)$$

$$H_{4,1} = \sum_{i=1}^N \tilde{\phi} \mathbf{f}(\mathbf{r}_i - \mathbf{r}_{N+1}) \frac{\nabla_i}{i}, \quad (13)$$

$$H_{4,2} = - \sum_{i=1}^N \tilde{\phi} \mathbf{f}(\mathbf{r}_i - \mathbf{r}_{N+1}) \frac{\nabla_{N+1}}{i}. \quad (14)$$

In (12), we inserted the vector potential  $\mathbf{A}$  in the symmetric gauge  $\mathbf{A}(\mathbf{r}) = \frac{1}{2} \begin{pmatrix} y \\ -x \end{pmatrix}$ . Since we have the translation invariance [3] of the Chern-Simons Green's function at the filling fraction  $\nu = 1/\tilde{\phi}$  it is possible to fix  $\mathbf{r} = 0$ . To calculate the asymptotic behaviour of the Chern-Simons Green's function, we use the cumulant theorem

$$\langle e^A \rangle = e^{\langle A \rangle_c + \frac{1}{2} \langle A^2 \rangle_c + \frac{1}{3!} \langle A^3 \rangle_c + \dots} \quad (15)$$

The 'connected' expectation values are defined by  $\langle A \rangle_c = \langle A \rangle$ ,  $\langle A^2 \rangle_c = \langle (A - \langle A \rangle)^2 \rangle$ ,  $\langle A^3 \rangle_c = \langle (A - \langle A \rangle)^3 \rangle, \dots$

We now split the Chern-Simons Green's functions in two parts  $G(\mathbf{r}, t, \mathbf{r}', t') = G^1(\mathbf{r}, t, \mathbf{r}', t') + G^2(\mathbf{r}, t, \mathbf{r}', t')$ .  $G^1(\mathbf{r}, t, \mathbf{r}', t')$  is the relevant term for  $\mathbf{r} \rightarrow \mathbf{r}'$  and has the form  $h(t-t', \nabla_{\mathbf{r}'}) \exp[-(t-t')1/2[-i\nabla_{\mathbf{r}'} + \mathbf{A}(\mathbf{r}')]^2] \delta(\mathbf{r} - \mathbf{r}')$ .  $h(t-t', \nabla_{\mathbf{r}'})$  is a power series in the derivations in  $\mathbf{r}'$  with prefactors in  $(t-t')$ .  $G^2(\mathbf{r}, t, \mathbf{r}', t')$  is a regular function (not of a distributional form). In the following, we will only calculate the constant coefficient of the power series  $h$  in  $\nabla_{\mathbf{r}'}$ . We will see in subsection D that  $G$  consists only on  $G^2$  for the time ordering  $t-t' < 0$ . Since the ground state energy can be calculated from the Green's function with this ordering [18], we are especially interested in that case. A second reason for calculating only the constant coefficient of the power series  $h(t-t', \nabla_{\mathbf{r}'})$  is that we are primary interested in a comparison of an approximative Green's function (see Sect. 3) with the exact Green's function for small  $|t-t'|$ . In a perturbational calculation of physical quantities one commonly has to calculate integrals of a function of Green's functions. The main contribution of the Green's functions to these integrals is given for small  $|t-t'|$  (the Green's function tends to zero for large  $|t-t'|$ ). Thus the approximated Green's function should be rather exact in this range. We will see at the end of this subsection that the higher order powers of  $h$  in  $\nabla_{\mathbf{r}'}$  contribute only on order  $O(|t-t'|^2)$ .

In this subsection, we will calculate  $G^1(\mathbf{r}, t, \mathbf{r}', t')$ . With the the help of a binomial expansion of the products as well as the form of the normalized lowest Landau level wave functions (5), we get

$$\begin{aligned} \langle u_p | \left[ -1 + \left( 2\mathbf{f}(\mathbf{r}) \frac{\nabla_{\mathbf{r}}}{i} \right)^2 \right]^{n'} \left[ -1 + 2\mathbf{f}(\mathbf{r}) \frac{\nabla_{\mathbf{r}}}{i} \right]^n | u_p \rangle &= \delta_{m,0} \delta_{n,0} \delta_{n',0} \\ &\times \left[ 2\mathbf{f}(\mathbf{r}) \frac{\nabla_{\mathbf{r}}}{i} \right]^l \frac{1}{r^m} \left[ \frac{\mathbf{r}}{r} \nabla_{\mathbf{r}} \right]^m | u_p \rangle = \delta_{m,0} \delta_{n,0} \delta_{n',0} \\ &+ \frac{1}{p} \left[ -\delta_{m,2} \frac{1}{4} \delta_{n,0} \delta_{n',0} \right. \\ &+ \delta_{m,1} \left\{ \frac{1}{2} \delta_{n',0} \delta_{n,1} + \delta_{n',1} \delta_{n,0} + \frac{1}{2} l \delta_{n,0} \delta_{n',0} \right\} \\ &+ \delta_{m,0} \left\{ \delta_{n',2} 4 \delta_{n,0} + \delta_{n',1} (2\delta_{n,1} + (2l+1)\delta_{n,0}) \right. \\ &\left. + \delta_{n',0} \left( \delta_{n,2} + l \delta_{n,1} + \frac{(l^2-l)}{2} \delta_{n,0} \right) \right\} \left. \right] + O\left(\frac{1}{p^2}\right). \end{aligned} \quad (16)$$

This relation is useful for getting the cumulant expectation values (15) in an easy way. For calculating the average val-

ues in (6) we consider in the following only terms behaving like  $\sum 1/p$  (this sum diverges for  $A \rightarrow \infty$ ). Terms of the form  $\sum O(1/p^2)$  are convergent. At first, we neglect  $H_0$  in  $H_{N+1} + \overline{H}_{N+1}$ . So we have to calculate the connected average values of the form  $\langle |(H_1 + H_2 + H_3 + H_{4,1} + H_{4,2})^n| \rangle_c$ . By neglecting finite terms of the form  $\sum O(1/p^2)$  we get with the help of equation (16) the following diverging terms:

$$\begin{aligned} \langle u_{0,\mathbf{k}} | H_2 | u_{0,\mathbf{k}} \rangle_c &= -\frac{1}{4} \tilde{\phi}^2 \sum_{u_p, u_{p+1} \in u_{0,\mathbf{k}}} \frac{1}{p}, \\ \langle u_{0,\mathbf{k}} | H_1 | u_{0,\mathbf{k}} \rangle_c &= \frac{1}{2} \tilde{\phi}^2 \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}, \\ \langle u_{0,\mathbf{k}} | (H_3 + H_{4,1})^n | u_{0,\mathbf{k}} \rangle_c &= \delta_{n,2} \frac{1}{4} \tilde{\phi}^2 \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}. \end{aligned} \quad (17)$$

We see from the first term of this equation that we have also summations over nearest neighbour  $p$ . Now we take into account the operator  $H_0$  in the calculation of the connected average values. For doing this, we will use the Campbell-Hausdorff formula

$$e^C = e^B \cdot e^A, \quad (18)$$

with

$$\begin{aligned} B = (C - A) + \frac{1}{2}[A, C] + \frac{1}{12}[C + A, [A, C]] \\ - \frac{1}{24}[A, [C, [A, C]]] + O([\cdot, [\cdot, [\cdot, \cdot]]]) \end{aligned} \quad (19)$$

We now fix  $C = -(H_{N+1} + \overline{H}_{N+1})\tau$  and  $A = -H_0\tau$  in equation (18). With the help of the Campbell-Hausdorff formula (18) and the cumulant theorem (15) one can discuss the connected average values in  $B$ . This results in a power series in  $\tau = (t-t')$ . We will show later on in this subsection that this power series will not terminate. Because of the enormous effort of calculation, we will limit ourselves to the concrete calculation of the coefficients up to the order  $\tau^3$ . It will be the aim of a later publication to make predictions of these coefficients for higher powers of  $\tau$  and further on the behaviour of this power series for large  $\tau$ .

The relevant cumulant expectation values for calculating the coefficients up to the order  $\tau^3$  are shown in Appendix A. With the help of these cumulant expectation values and the Campbell-Hausdorff formula (18), we get for the asymptotics of the Green's function for  $\tau > 0$ :

$$\begin{aligned} G^1(\mathbf{r}, \tau)_{\tau > 0} &= \sum_{u_{0,\mathbf{k}} \in \nu_{1/\tilde{\phi}}} \exp \left[ \tilde{\phi}^2 \left\{ \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p} \left( -\frac{1}{2}\tau + \frac{1}{8}\tau^2 \right. \right. \right. \\ &\quad \left. \left. - \frac{1}{48}\tau^3 + O(\tau^4) \right) \right\} + O\left(\frac{1}{p^2}\right) \right] \\ &+ \sum_{u_p, u_{p+1} \in u_{0,\mathbf{k}}} \frac{1}{p} \left( \frac{1}{4}\tau - \frac{1}{48}\tau^3 + O(\tau^4) + O\left(\frac{1}{p^2}\right) \right) \left. \right] \frac{1}{\sum_{u_{0,\mathbf{k}} \in \nu_{1/\tilde{\phi}}} 1} \\ &\times \exp \left[ -\tau \frac{1}{2} [-i\nabla_{\mathbf{r}} + \mathbf{A}(\mathbf{r})]^2 \right] \delta(\mathbf{r}). \end{aligned} \quad (20)$$

We see from equation (20) that the cumulant expectation values up to the order  $\tau^3$  behave like  $\sum_{u_p \in u_{0,\mathbf{k}}} O(1/p) + \sum_{u_p, u_{p+1} \in u_{0,\mathbf{k}}} O(1/p)$ . In the following, we will show that equation (20) is correct for all orders of  $\tau$ . For doing this, we consider the operator  $H_{0,1}$  in polar coordinates:

$$H_{0,1} = \sum_{i=1}^N \frac{1}{2} \left[ -\frac{1}{r_i} \partial_{r_i} - \partial_{r_i}^2 + \left( -\frac{1}{r_i^2} \partial_{\phi_i}^2 + i \partial_{\phi_i} + \frac{r_i^2}{4} \right) \right]. \quad (21)$$

The expectation value of the commutator of the operator  $(1/r_i) \partial_{r_i}$  with the last three summands of (21) results in the first factor on the left hand side in (16). Furthermore, one sees that the second term of the product in (16) corresponds to  $(H_3 + H_{4,1})$ . Thus, we get from (16) and the form of  $H_{0,1}$  that by neglecting  $H_{0,2}$  and  $H_{4,2}$  in  $H_{N+1} + \overline{H}_{N+1}$  the cumulant expectation values of the Campbell-Hausdorff terms behave like  $\sum_{u_p \in u_{0,\mathbf{k}}} O(1/p) + \sum_{u_p, u_{p+1} \in u_{0,\mathbf{k}}} O(1/p)$ . In the following, we consider the cumulant expectation values of  $B$  (19) containing at least one of the terms  $H_{0,2}$ ,  $H_{4,2}$ . It is clear by having more than two terms  $H_{4,2}$  in the cumulant expectation value, we get as a result a convergent sum  $\sum O(1/p^2)$ . The number of operators  $H_{0,2}$  is arbitrary. Thus, we can limit our consideration to one or two operators  $H_{4,2}$  in the cumulant expectation value. Because of the simple structure of  $H_{N+1} + \overline{H}_{N+1}$  we get as a result of the cumulant expectation value a term of the form  $\sum_{u_p \in u_{0,\mathbf{k}}} (a_1/p + O(1/p^2)) + \sum_{u_p, u_{p+1} \in u_{0,\mathbf{k}}} (a_2/p + O(1/p^2))$ . Here  $a_1$ ,  $a_2$  are real coefficients. Summarizing, we see that the exponent of the Green's functions  $G^1(0, \tau)_{\tau > 0}$  behaves like (20).

## 2.2 The asymptotic behaviour of the Chern-Simons Green's function $G(\mathbf{r}, \tau)_{\tau > 0}$ for $\mathbf{r} \neq 0$

In this section, we calculate  $G(\mathbf{r}, \tau)_{\tau > 0}$  for  $\tau > 0$  and  $\mathbf{r} \neq 0$ . Because of the additional phase factor, we get from equation (6) that the calculation of the asymptotic behaviour of the Green's function for  $\mathbf{r} \neq 0$  is substantially more difficult than for  $\mathbf{r} = 0$ . To handle this phase factor we define

$$\begin{aligned} \exp[M[N']] &:= \exp \left[ i \tilde{\phi} \left( \sum_{i=1}^{N'} \alpha(\mathbf{r}_i - \mathbf{r}') - \alpha(\mathbf{r}_i - \mathbf{r}) \right) \right] \\ &= \exp \left[ i \tilde{\phi} \left( \sum_{i=1}^{N'} \mathbf{f}(\mathbf{r}_i - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}') \right. \right. \\ &\quad \left. \left. + (\mathbf{f}(\mathbf{r}_i - \mathbf{r}') \cdot \mathbf{e}_x) (\mathbf{e}_y \cdot \mathbf{f}(\mathbf{r}_i - \mathbf{r}')) \right. \right. \\ &\quad \left. \left. \times ((\mathbf{r} - \mathbf{r}') \cdot \mathbf{e}_x)^2 - (\mathbf{e}_y \cdot (\mathbf{r} - \mathbf{r}'))^2 \right) \right. \\ &\quad \left. + O \left( \frac{1}{|\mathbf{r}_i - \mathbf{r}'|^3} \right) \right], \end{aligned} \quad (22)$$

with  $0 \leq N' \leq N$ .  $\mathbf{e}_x$  and  $\mathbf{e}_y$ , respectively, are unit vectors in the  $x$ - and  $y$ -direction, respectively. When calculating

the Green's function (6), we get one term which is proportional to  $\delta(\mathbf{r} - \mathbf{r}')$ . This term was calculated in the last subsection. Additionally, we get

$$\begin{aligned} G^2(\mathbf{r}, t; \mathbf{r}', t') &= - \frac{e^{((t-t')-\beta)(\frac{1}{2}-\mu)N}}{Z} \\ &\times \sum_{u_{0,\mathbf{k}} \in \nu_{1/\tilde{\phi}}} \sum_{u_L, u_R \in u_{0,\mathbf{k}}} \langle V_L[u_{0,\mathbf{k}}] | e^{M[N-1]} e^{-(t-t')H_2} | V_R[u_{0,\mathbf{k}}] \rangle \\ &\times \left( e^{-(t-t')\frac{1}{2}} [-i \nabla_{\mathbf{r}'} + \mathbf{A}(\mathbf{r}') - \tilde{\phi} \mathbf{f}(\mathbf{r}' - \mathbf{r})]^2 u_L(\mathbf{r}') \right) \\ &\times \left( e^{-(t-t')\frac{1}{2}} [-i \nabla_{\mathbf{r}} + \mathbf{A}(\mathbf{r}) + \tilde{\phi} \mathbf{f}(\mathbf{r} - \mathbf{r}')]^2 u_R(\mathbf{r}) \right) \end{aligned} \quad (23)$$

with

$$H_2 = \sum_{i=1}^{N-1} \frac{1}{2} \left[ -i \nabla_{r_i} + \mathbf{A}(\mathbf{r}_i) + \tilde{\phi} \mathbf{f}(\mathbf{r}_i - \mathbf{r}') \right]^2 \quad (24)$$

$V_p[u_{0,\mathbf{k}}]$  is defined by

$$V_p[u_{0,\mathbf{k}}] = u_{0,\mathbf{k}'} \sum_{i=1}^N (-1)^{i+1} \delta_{p,k_i}. \quad (25)$$

$\mathbf{k}' \in \mathbb{R}^{N-1}$  is up to the entry which contains  $p$  given by the vector  $\mathbf{k}$ . In the following, we will transform (23) such that we can use the results of the last subsection. For doing this, we define the following operators:

$$\begin{aligned} O_\gamma &:= \gamma \left( e^{-\tau \frac{1}{2} [-i \nabla_{\mathbf{r}'} + \mathbf{A}(\mathbf{r}') - \tilde{\phi} \mathbf{f}(\mathbf{r}' - \mathbf{r})]^2} \right. \\ &\quad \left. \times \sum_{i=1}^N P[\mathbf{r}' - \mathbf{r}](\mathbf{r}_i) \delta(\mathbf{r}_i - \mathbf{r}) \right), \\ P[\mathbf{r}' - \mathbf{r}](\mathbf{r}_i) &:= 1 + (\mathbf{r}' - \mathbf{r}) \cdot \nabla_i + \frac{1}{2} \dots, \end{aligned} \quad (26)$$

$$H_M(\gamma, \tau) := \overline{H}_{N+1} - \frac{1}{\tau} O_\gamma - \frac{1}{\tau} M[N]. \quad (27)$$

The operator  $P[\mathbf{r}' - \mathbf{r}](\mathbf{r}_i)$  is a translation operator which yields by applying it to a function  $f(\mathbf{r}_i)$  the translated function  $f(\mathbf{r}_i + \mathbf{r}' - \mathbf{r})$ . Let us suppose that  $R$  is a power expansion of the operators  $O_\gamma$ ,  $M[N]$  and further on other operators. Then we define the normal ordering of the operator  $: R :$  such that all operators  $O_\gamma$  are at the left of the monomials and the following operators are the  $M[N]$ . With the help of the function

$$\begin{aligned} F_M[\gamma, \tau] &:= \exp \left[ -\tau \langle | : H_M(\gamma, \tau) : | \rangle_c \right. \\ &\quad \left. + \frac{1}{2} \tau^2 \langle | : H_M(\gamma, \tau)^2 : | \rangle_c + \dots \right], \end{aligned} \quad (28)$$

we get

$$\begin{aligned} G^2(\mathbf{r}, t; \mathbf{r}', t') &= - \frac{e^{(t-t'-\beta)(\frac{1}{2}-\mu)N}}{Z} \sum_{u_{0,\mathbf{k}} \in \nu_{1/\tilde{\phi}}} e^{-i \tilde{\phi} \alpha(\mathbf{r} - \mathbf{r}')} \\ &\times \frac{\partial}{\partial \gamma} F_M[\gamma, t - t'] \Big|_{\gamma=0}. \end{aligned} \quad (29)$$

The derivation with respect to  $\gamma$  of  $F_M[\gamma, t - t']$  at  $\gamma = 0$  is given by

$$\frac{\partial}{\partial \gamma} F_M[\gamma, \tau] \Big|_{\gamma=0} = F_M[0, \tau] \frac{\partial}{\partial \gamma} \left( -\tau \langle | : H_M(\gamma, \tau) : \rangle_c + \frac{1}{2} \tau^2 \langle | : H_M(\gamma, \tau)^2 : \rangle_c + \dots \right) \Big|_{\gamma=0}. \quad (30)$$

We now split  $\bar{H}_{N+1}$  (27) in its components

$$\bar{H}_{N+1} = \sum_{i=1}^N \left( \frac{1}{2} [-i \nabla_i + \mathbf{A}(\mathbf{r}_i)]^2 - \mu \right) + \frac{1}{2} \sum_{i=1}^N \tilde{\phi}^2 \mathbf{f}(\mathbf{r}_i - \mathbf{r}')^2 + \left( -N \frac{\tilde{\phi}}{2} + \sum_{i=1}^N \tilde{\phi} \mathbf{f}(\mathbf{r}_i - \mathbf{r}') \frac{\nabla_i}{i} \right). \quad (31)$$

As in the last subsection, we use the Campbell-Hausdorff formula (18) to isolate  $H_{0,1}$  from  $\bar{H}_{N+1}$  and  $1/2[-i \nabla_{r'} + \mathbf{A}(\mathbf{r}')]^2$  from  $1/2[-i \nabla_{r'} + \mathbf{A}(\mathbf{r}') - \tilde{\phi} \mathbf{f}(\mathbf{r}' - \mathbf{r})]^2$ . We now set  $\mathbf{r}' = 0$ . Then we can use the commutator analysis of appendix A and the results of the last subsection. First, we get that the second term of the product in (30) does not contain any terms of the form  $\sum 1/p$  (under the consideration that the  $l$ th derivative of  $u_p(\mathbf{r}')$  with respect to  $\mathbf{r}'$  is at the position  $\mathbf{r}' = 0$  zero for all  $p > l$ ). The first term in the product (30) is given by

$$G^2(\mathbf{r}, \tau)_{\tau>0} = \sum_{u_0, \mathbf{k} \in \nu_{1/2}} \exp \left[ \tilde{\phi}^2 \left\{ \sum_{u_p \in u_0, \mathbf{k}} \frac{1}{p} \left( -\frac{1}{4} r^2 - \frac{1}{4} \tau + \frac{1}{8} \tau^2 - \frac{1}{24} \tau^3 + O(\tau^4) \right) + O\left(\frac{1}{p^2}\right) + \sum_{u_p, u_{p+1} \in u_0, \mathbf{k}} \frac{1}{p} \frac{1}{4} r^2 + O\left(\frac{1}{p^2}\right) \right\} \right]. \quad (32)$$

We see from  $G^2(\mathbf{r}, \tau)_{\tau>0}$  that the  $\sum 1/p$  coefficients of the higher order  $\tau$  terms ( $\tau > 3$ ) have no  $\mathbf{r}$  dependence. Furthermore, one gets from the form of  $\bar{H}_{N+1}$  and the relation (16) that the higher order  $\tau$  terms have not to be zero.

### 2.3 Averaging of the Green's function with respect to the ground states

From the equations (20, 32), we have to calculate an expression of the following form

$$G(\mathbf{r}, \tau) = \sum_{u_0, \mathbf{k} \in \nu_{1/\tilde{\phi}}} \exp \left[ \sum_{u_p \in u_0, \mathbf{k}} (\tilde{\phi}^2 f_1(r, \tau)) \frac{1}{p} + O\left(\frac{1}{p^2}\right) + \sum_{u_p, u_{p+1} \in u_0, \mathbf{k}} (\tilde{\phi}^2 f_2(r, \tau)) \frac{1}{p} + O\left(\frac{1}{p^2}\right) \right] \frac{1}{\sum_{u_0, \mathbf{k} \in \nu_{1/2}} 1}. \quad (33)$$

$f_1(r, \tau)$  and  $f_2(r, \tau)$  are real functions of  $(r, \tau)$ .

In the following, we will show that the asymptotic behaviour of the result is given by

$$G(\mathbf{r}, \tau) = \exp \left[ -2 \left( \tilde{\phi} f_1(r, \tau) + f_2(r, \tau) \right) \log(c) + O(1) \right]. \quad (34)$$

Here  $c$  is proportional to  $1/\sqrt{A}$ . So we get  $\log(c) = -\frac{1}{2} \log(N) + O(1)$ . With the help of the expression

$$E[u_0, \mathbf{k}] = \sum_{u_p \in u_0, \mathbf{k}} (\tilde{\phi}^2 f_1(r, \tau)) \left( \frac{1}{p} - \frac{1}{N \tilde{\phi}} (\log(N) + \mathcal{C}) \right) + O\left(\frac{1}{p^2}\right) + \sum_{u_p, u_{p+1} \in u_0, \mathbf{k}} (\tilde{\phi}^2 f_2(r, \tau)) \left( \frac{1}{p} - \frac{1}{N \tilde{\phi}^2} (\log(N) + \mathcal{C}) \right) + O\left(\frac{1}{p^2}\right), \quad (35)$$

the asymptotics (34) is correct if we show that the following expression  $K$  is finite for  $A \rightarrow \infty$ :

$$K = \sum_{u_0, \mathbf{k} \in \nu_{1/\tilde{\phi}}} \exp[E[u_0, \mathbf{k}]] \frac{1}{\sum_{u_0, \mathbf{k} \in \tilde{\phi}} 1}. \quad (36)$$

$\mathcal{C}$  is the Euler number which is defined by  $\sum_{p=1}^N 1/p = \log(N) + \mathcal{C} + O(1/N)$ . We will determine  $K$  by the calculation of the moments  $M_n$  of the operator  $\exp[E(u_0, \mathbf{k})]$ . The moments are defined by  $M_n = \sum_{u_0, \mathbf{k} \in \nu_{1/2}} (E[u_0, \mathbf{k}])^n \cdot 1/(\sum_{u_0, \mathbf{k} \in \nu_{1/2}} 1)$ . From this definition we obtain  $M_0 = 1$ .

For calculating  $M_1$ , we use the following transformation

$$\frac{1}{\sum_{u_0, \mathbf{k} \in \nu_{1/\tilde{\phi}}} 1} \sum_{u_0, \mathbf{k} \in \nu_{1/2}} \sum_{u_p \in u_0, \mathbf{k}} \frac{1}{p} = \frac{1}{\sum_{u_0, \mathbf{k} \in \nu_{1/\tilde{\phi}}} 1} \sum_{p=1}^N \frac{1}{p} \left( \frac{N-1}{\tilde{\phi}} - 1 \right) = \frac{1}{\tilde{\phi}} \left( \log(N) + \mathcal{C} + O\left(\frac{1}{N}\right) \right), \quad (37)$$

$$\frac{1}{\sum_{u_0, \mathbf{k} \in \nu_{1/\tilde{\phi}}} 1} \sum_{u_0, \mathbf{k} \in \nu_{1/2}} \sum_{u_p, u_{p+1} \in u_0, \mathbf{k}} \frac{1}{p} = \frac{1}{\sum_{u_0, \mathbf{k} \in \nu_{1/\tilde{\phi}}} 1} \sum_{p=1}^N \frac{1}{p} \left( \frac{N-2}{\tilde{\phi}} - 2 \right) = \frac{1}{\tilde{\phi}^2} \left( \log(N) + \mathcal{C} + O\left(\frac{1}{N}\right) \right). \quad (38)$$

Thus, we get that  $M_1$  is finite.

For calculating the higher moments, we will show at first that  $A_{(M_1+M_2)}$  is finite.  $A_{(M_1+M_2)}$  is defined by

$$A_{(M_1+M_2)} = \lim_{N \rightarrow \infty} \sum_{u_0, \mathbf{k} \in \nu_{1/\tilde{\phi}}} \left[ \prod_{i'=1}^{M_2} \left( \sum_{u_p \in u_0, \mathbf{k}} \frac{1}{p^{m_{i'}}} \right)^{n_{i'}} \right] \times \prod_{i=1}^{M_1} \left( \sum_{u_p \in u_0, \mathbf{k}} \left( \frac{1}{p} - \frac{1}{N \tilde{\phi}} (\log(N) + \mathcal{C}) \right) \frac{1}{p^{m_i}} \right) \frac{1}{\sum_{u_0, \mathbf{k} \in \nu_{1/\tilde{\phi}}} 1}. \quad (39)$$

We are interested in  $A_{(M_1+M_2)}$  for the values  $n_i \geq 1, m_{i'} \geq 2, m_i \geq 2$  or  $m_i = 0$ . To show the finiteness of  $A_{(M_1+M_2)}$  we carry out a proof by induction.

Because of the equations (37, 38) and the finiteness of  $\sum_{u_p \in u_0, k \in \nu_{1/2}} 1/p^n$  for  $n \geq 2$  we get that  $A_1$  is finite. For  $J > 1$  we can split  $A_J$  in the following two terms:

$$A_J = \lim_{N \rightarrow \infty} \sum_{u_0, k \in \nu_{1/2}} \left[ \sum_{\substack{u_{p_i}, u_{p_{i'}} \in u_0, k \\ p_i \neq p_{i'}}} \prod_{i'=1}^{M_2} \left( \frac{1}{p_{i'}} \right) \right. \quad (40) \\ \times \prod_{i=1}^{M_1} \left( \frac{1}{p_i} - \frac{1}{N\tilde{\phi}} (\log(N) + C) \right)^{n_i} \\ \left. \times \left( \frac{1}{p_i} \right) \frac{1}{\sum_{u_0, k \in \nu_{1/2}} 1} \right] + A_{J-1}.$$

By  $p_i \neq p_{i'}$  we mean that all  $p_i, p_{i'}$  are different. Due to the induction assumption, we only have to discuss the first term  $A_{J,1}$  in  $A_J$ . With the help of the induction assumption we get for  $A_{J,1}$  as in the calculation of  $M_1$

$$A_{J,1} = \frac{1}{\tilde{\phi}^{|M_1|+|M_2|}} \prod_{i=1}^{M_1} \prod_{i'=1}^{M_2} \sum_{p_i, p_{i'}=1}^N \left[ \left( \frac{1}{p_{i'}} \right) \quad (41) \right. \\ \left. \left( \frac{1}{p_i} - \frac{1}{N\tilde{\phi}} (\log(N) + C) \right)^{n_i} \left( \frac{1}{p_i} \right) \right] + \text{finite}.$$

From this we see that  $A_{J,1}$  is finite. Thus, we get the finiteness of  $A_{(M_1+M_2)}$  for  $N \rightarrow \infty$ .

Summarizing, we obtain the finiteness of  $M_n$  for every  $n$  if  $f_2(r, \tau) = 0$ . It is easy to generalize the considerations above to the case of  $f_2(r, \tau) \neq 0$ . Thus the Chern-Simons Green's function  $G(\mathbf{r}, \tau)_{\tau > 0}$  has the form of equation (34). With the help of (20) and (32) we get for the asymptotic behaviour of the Green's function  $G(\mathbf{r}, \tau)_{\tau > 0} = G^1(\mathbf{r}, \tau)_{\tau > 0} + G^2(\mathbf{r}, \tau)_{\tau > 0}$

$$G^1(\mathbf{r}, \tau)_{\tau > 0} \sim \exp \left[ -2\tilde{\phi} \log(c) \left\{ - \left( \frac{1}{2} - \frac{1}{4\tilde{\phi}} \right) \frac{B}{m} \tau \right. \right. \\ \left. \left. + \frac{1}{8} \left( \frac{B}{m} \tau \right)^2 - \left( \frac{1}{48} + \frac{1}{48\tilde{\phi}} \right) \left( \frac{B}{m} \tau \right)^3 + O(\tau^4) \right\} \right] \\ \times \exp \left[ -\tau \frac{1}{2m} [-i\nabla_r + \mathbf{A}(\mathbf{r})]^2 \right] \delta(\mathbf{r}) \quad (42)$$

and

$$G^2(\mathbf{r}, \tau)_{\tau > 0} \sim \exp \left[ -2\tilde{\phi} \log(c) \left\{ - \left( \frac{1}{4} - \frac{1}{4\tilde{\phi}} \right) \frac{B}{m} r^2 \right. \right. \\ \left. \left. - \frac{1}{4} \frac{B}{m} \tau + \frac{1}{8} \left( \frac{B}{m} \tau \right)^2 - \frac{1}{24} \left( \frac{B}{m} \tau \right)^3 + O(\tau^4) \right\} \right]. \quad (43)$$

## 2.4 The asymptotic behaviour of the Chern-Simons Green's function $G(\mathbf{r}, \tau)_{\tau < 0}$

In this subsection, we calculate the Chern-Simons Green's function  $G(\mathbf{r}, \tau)_{\tau < 0}$  for  $\tau < 0$ . At first, we define a wave function originating from a  $\nu = 1/\tilde{\phi}$  wave function in which a particle at position  $\mathbf{r}$  is annihilated. This wave function is given by

$$u'_{0,k}(\mathbf{r}_1, \dots, \mathbf{r}_{N-1}; \mathbf{r}) := \sum_{i=1}^N \int d\mathbf{r}_i u'_{0,k}(\mathbf{r}_1, \dots, \mathbf{r}_N) \delta(\mathbf{r}_i - \mathbf{r}) \quad (44)$$

(in the definition of equation (44) we implicitly carried out a renaming of the indices). Similar to our calculation in subsection A we get for  $t - t' < 0$  from the definition (3)

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{e^{((t'-t)-\beta)(\frac{1}{2}-\mu)N}}{Z} \quad (45) \\ \times \sum_{u_0, k} \left\langle u'_{0,k}(\mathbf{r}_1, \dots, \mathbf{r}_{N-1}; \mathbf{r}) \right| \\ \times \exp \left[ -i\tilde{\phi} \left( \sum_{i=1}^{N-1} \alpha(\mathbf{r}_i - \mathbf{r}) - \alpha(\mathbf{r}_i - \mathbf{r}') \right) \right] \\ \times \exp \left[ -(t' - t) \bar{H}_N \right] \left| u'_{0,k}(\mathbf{r}_1, \dots, \mathbf{r}_{N-1}; \mathbf{r}') \right\rangle$$

with the Hamilton operator

$$\bar{H}_N = \sum_{i=1}^{N-1} \frac{1}{2} \left[ -i\nabla_i + \mathbf{A}(\mathbf{r}_i) + \tilde{\phi} \mathbf{f}(\mathbf{r}_i - \mathbf{r}') \right]^2. \quad (46)$$

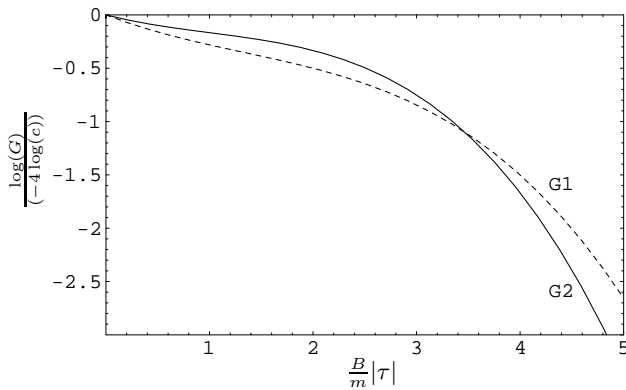
As in the last subsections, we split  $\bar{H}_N$  in its components

$$\bar{H}_N = \sum_{i=1}^{N-1} \left( \frac{1}{2} [-i\nabla_i + \mathbf{A}(\mathbf{r}_i)]^2 - \mu \right) + \frac{1}{2} \sum_{i=1}^{N-1} \tilde{\phi}^2 \mathbf{f}(\mathbf{r}_i - \mathbf{r}')^2 \\ + \left( -(N-1) \frac{\tilde{\phi}}{2} + \sum_{i=1}^{N-1} \tilde{\phi} \mathbf{f}(\mathbf{r}_i - \mathbf{r}') \frac{\nabla_i}{i} \right). \quad (47)$$

Now we can use the commutator analysis of Appendix A and the results of the last subsections to get for the Chern-Simons Green's function  $G(\mathbf{r}, \tau)_{\tau < 0}$

$$G(\mathbf{r}, \tau)_{\tau < 0} \sim \exp \left[ -2\tilde{\phi} \log(c) \left\{ - \left( \frac{1}{4} - \frac{1}{4\tilde{\phi}} \right) \frac{B}{m} r^2 \right. \right. \\ \left. \left. + \frac{1}{4} \frac{B}{m} \tau + \frac{1}{8} \left( \frac{B}{m} \tau \right)^2 + \frac{1}{24} \left( \frac{B}{m} \tau \right)^3 + O(\tau^4) \right\} \right]. \quad (48)$$

By taking the limit  $A \rightarrow \infty$  we get from the equations (42, 43) and (48) that the Chern-Simons Green's function  $G(\mathbf{r}, \tau)$  vanishes exponentially for  $\tau > 0$  as well as for  $\tau < 0$ . This is illustrated in Figure 1 where we show the function  $\log(G)/(-4\log(c))$  for the  $\nu = 1/2$  system.  $G$  is either the first factor in (42) for  $\mathbf{r} = 0$  or  $G = G(0, |\tau|)_{\tau > 0} = G(0, |\tau|)_{\tau < 0}$  (43, 48).



**Fig. 1.**  $\log(G)/(-4\log(c))$  for the  $\nu = 1/2$  system where  $G$  is either the first factor in (42) for  $\mathbf{r} = 0$  (dashed line), or  $G = G^2(0, |\tau|)_{\tau>0} = G(0, |\tau|)_{\tau<0}$  (solid line).

The vanishing of the Green's function can be explained by the boundedness of the Green's function (easily obtained from the definition (3)). It is an interesting question if the  $\log(c)$  behaviour is also true when considering also higher  $\tau$  terms in the concrete calculation of the Green's function. Since the number of summands of the Campbell-Hausdorff formula grows very fast for large  $\tau$  we think there is no easy answer to this question. As mentioned above, we are primarily interested in this paper in the Green's function for small  $\tau$ . So instead of considering further this question, we investigate the asymptotics of the Chern-Simons Green's function taking into consideration the Coulomb interaction.

## 2.5 The asymptotics of the Green's function taking into consideration the Coulomb interaction

In the case of the Chern-Simons theory taking into consideration the Coulomb interaction the ground state  $u_0$  consists of a superposition of Slater determinants which consists of Landau wave functions of the lowest as well as of higher Landau levels. In the following, we make the approximation that we consider in  $u_0$  only that part which consists of wave functions in the lowest Landau level. This restriction on the wave function is a standard approximation to calculate for example the ground state energy or the effective mass of the composite fermions [16]. The ground state wave function is given by  $u_0 = \sum_{\mathbf{p} \in \mathbb{N}_0^N} c_{\mathbf{p}} S[u_{p_1}, u_{p_2}, \dots]$ . The coefficients  $c_{\mathbf{p}}$  underlie the restriction  $\sum_{\mathbf{p} \in \mathbb{N}_0^N} |c_{\mathbf{p}}|^2 = 1$ . This is a result of the normalization of  $u_0$ . To get insight into the Chern-Simons Green's function taking into account the Coulomb interaction, we can transform the Green's function similar to (6) and (45) with an additional sum over the Slater determinants of  $u_0$ . With the help of the Campbell-Hausdorff formula we separate similarly to the last subsections the Hamilton operator  $H_0$  from the rest of the operators in  $H_{N+1} + \overline{H}_{N+1}$ ,  $\overline{H}_{N+1}$  or  $\overline{H}_N$  ( $H_0$  also consists of the Coulomb part of the Hamilton operator). Suppose at first the restriction that the two Slater determinants which are the ingredients of the average values in (6) and (45) are

not in agreement in their Landau wave functions  $u_{p_i}$  for  $i = 1, \dots, N_C$  with  $N_C \leq N$ . Furthermore, suppose that  $N - N_C$  is finite for  $A \rightarrow \infty$ . Then we get that the average value of  $e^B$  (19) with respect to these two Slater determinants behaves as  $\lesssim \prod_{i=1, \dots, N_C} 1/p_i^{1/4} \lesssim e^{-1/4 N_C}$  (the Coulomb operator in a scalar product between two Slater determinants of Landau wave functions  $u_{p_j}$   $j = 1, \dots, 4$  scales like  $\sim \text{Min}[\sqrt{1/p_j}]$ . Here  $\text{Min}[\cdot]$  is the minimum of its argument). Since  $N_C \propto A$  for  $A \rightarrow \infty$ , we see that the average value with respect to the two Slater determinants vanishes exponentially for  $A \rightarrow \infty$ .

In the following, we consider the case of no restrictions on the variables  $N$ ,  $N_C$ . Then we get from the equations (15, 33) that the asymptotic form of the summands of the Green's function behaves as  $\lesssim e^{-1/4 N_C} e^{-f(r, \tau, N, e^2) \log(N - N_C)} f(r, \tau, N, e^2)$  is given by  $f(r, \tau, N, e^2) = f(r, \tau) (\sum_{j=1 \dots N} (1/p_j) - \sum_{i=1 \dots N_C} (1/p_i)) / \log(N - N_C)$  (we neglect the nearest neighbour sum in (33)). Furthermore, we have  $0 \leq \lim_{N \rightarrow \infty} f(r, \tau, N, e^2) \leq f(r, \tau)$  ( $\lim_{N \rightarrow \infty} f(r, \tau, N, e^2)$  has not to be convergent). Thus, we get that the maximum of the summands of the Green's function is given by equal Slater determinants of which the average values (6) and (45) are built. Summarizing, we get also in the case of the Chern-Simons theory taking into account the Coulomb interaction the asymptotics (42, 43) and (48) of the Green's function, whereby we have to replace  $\log(c)$  by  $V(e^2, c) \log(c)$  in these formulas.  $V(e^2, c)$  is a function of the Coulomb coupling constant  $e^2$  with  $0 \leq \lim_{c \rightarrow 0} V(e^2, c) \leq \tilde{\phi}$  and  $\lim_{c \rightarrow 0} V(0, c) = 1$ . In principle  $V(e^2, c)$  may have the limit  $\lim_{c \rightarrow 0} V(e^2, c) = 0$  for some  $e^2$ . Then we get that the Green's function does not vanish exponentially for  $A \rightarrow \infty$ . Moreover  $\lim_{c \rightarrow 0} V(e^2, c)$  does not have to be continuous at  $e^2 = 0$  for temperature  $T = 0$ . By considering the Green's function for  $T > 0$  (and limit the Hilbert space to the lowest Landau level) we get for the asymptotics of the Green's function the equations (42, 43) and (48) with the replacement  $\log(c)$  by  $V(e^2, c) \log(c)$  where  $V(e^2, c)$  is a function of the temperature. For  $T > 0$  the limit  $\lim_{c \rightarrow 0} V(e^2, c)$  has to be continuous at  $e^2 = 0$  with  $\lim_{e^2 \rightarrow 0} \lim_{c \rightarrow 0} V(e^2, c) = 1$ . Thus also in the case of the Chern-Simons theory taking into consideration the Coulomb interaction, we obtain an asymptotic behaviour of the Green's function which vanishes for  $A \rightarrow \infty$ .

We should remark a consequence of the asymptotic behaviour of the Green's function. By inserting between the creation and the annihilation operator in (3) a complete set of eigen functions of the Hamiltonian  $H_{CS}$ , we get that the overlap between the wave functions  $A[u_0(\mathbf{r}_1, \dots, \mathbf{r}_N), \delta(\mathbf{r}_{N+1} - \mathbf{r})]$  or  $\sum_{i=1}^N \int d\mathbf{r}_i u_0(\mathbf{r}_1, \dots, \mathbf{r}_N) \delta(\mathbf{r}_i - \mathbf{r})$ , respectively, and the eigen functions of the Hamilton operator  $H_{CS}$  with  $N + 1$  or  $N - 1$  particles, respectively, vanishes for  $A \rightarrow \infty$ .



### 3 The Green's functions in the Hartree-Fock approximation as well as in the RPA

When calculating physical quantities perturbatively the mean field Green's function of the perturbation theory should have a similar form as the exact Green's function. In the following, we compare the asymptotic behaviour of the Green's function in the Hartree-Fock approximation as well as in the RPA with the exact Green's function to make a step towards to this perturbation theory. We will show that in contrast to the RPA Green's function the Hartree-Fock Green's function has a similar asymptotic behaviour as the exact Green's function. Because in a perturbation theory with auxiliary fields one usually takes as a starting point a Green's function [18] which is determined self consistently, we will calculate the self consistent Hartree-Fock Green's function and compare the calculated density of this Green's function with the exact particle density. Since the Chern-Simons interaction is the reason for the asymptotic vanishing form of the Chern-Simons Green's function for  $A \rightarrow \infty$ , we neglect at first the Coulomb interaction.

It is easy to calculate the Hartree-Fock approximation of the Green's function of the Chern-Simons Hamiltonian (2) for  $A \rightarrow \infty$ . One gets

$$\begin{aligned} \Sigma^{\text{HF}}(q, k_{\text{F}}) &= \frac{\tilde{\phi}^2 k_{\text{F}}^2}{4 m} \left[ \log(4c^2) \right. \\ &\quad \left. - \log \left( 2\sqrt{(k_{\text{F}}^2 + q^2 + c^2)^2 - 4q^2 k_{\text{F}}^2} + 2k_{\text{F}}^2 + 2(c^2 - q^2) \right) \right] \\ &\quad - \frac{\tilde{\phi}^2 k_{\text{F}}^2}{4 m} \log \left( \frac{c}{k_{\text{F}}} \right) + \Sigma^f(q, k_{\text{F}}). \end{aligned} \quad (49)$$

As in the last section  $c$  is an impulse cut off in the infrared region. To get  $\Sigma^{\text{HF}}(q, k_{\text{F}})$  we calculated all Hartree-Fock Feynman diagrams by inserting the interaction free Green's function  $G(\mathbf{q}, \omega) = -1/(\mathrm{i}\omega - q^2/(2m) + \mu)$ . For this Green's function the Fermi momentum  $k_{\text{F}}$  is given by  $k_{\text{F}} = \sqrt{2m\mu}$ . By pairing the outer creation and annihilation operator in (2) we get an effective two particle operator which scales in the momentum like  $\sim 1/q^2$ . The first summand in (49) represents the exchange diagram of this vertex. The rest of the diagrams which diverge for  $c \rightarrow 0$  are given by the second summand. This term is given by a diagram of the Hartree form. The last term  $\Sigma^f$  is finite for  $c \rightarrow 0$ . The Hartree-Fock Green's function was also calculated by Sitko and Jacak [12] before. Carrying out the limit  $c \rightarrow 0$  in (49) we get

$$\begin{aligned} \Sigma^{\text{HF}}(q, k_{\text{F}}) &= \\ &\begin{cases} \frac{\tilde{\phi}^2 k_{\text{F}}^2}{4 m} \left[ \log \left( \frac{c}{k_{\text{F}}} \right) - \log \left( \frac{k_{\text{F}}^2 - q^2}{k_{\text{F}}^2} \right) \right] + \Sigma^f(q, k_{\text{F}}) & q < k_{\text{F}} \\ -\frac{\tilde{\phi}^2 k_{\text{F}}^2}{4 m} \left[ \log \left( \frac{c}{k_{\text{F}}} \right) - \log \left( \frac{q^2 - k_{\text{F}}^2}{q^2} \right) \right] + \Sigma^f(q, k_{\text{F}}) & q > k_{\text{F}} \\ \Sigma^f(k_{\text{F}}, k_{\text{F}}) & q = k_{\text{F}}. \end{cases} \end{aligned} \quad (50)$$

Thus, we see that  $\Sigma^{\text{HF}}$  is finite on the Fermi surface ( $q = k_{\text{F}}$ ). This is accomplished by the changing of the sign

of the  $\log(c)$  singularity on the Fermi surface. We now calculate the Fourier (time) transform of the Hartree-Fock Green's function (the Green's function which includes the Hartree-Fock self energy  $\Sigma^{\text{HF}}$ ). In the leading  $c$  order, we get for this Green's function

$$\begin{aligned} &\frac{1}{2\pi} \int d\omega G^{\text{HF}}(q, \omega) e^{-\mathrm{i}\omega\tau} \\ &= -e^{[-(\frac{q^2}{2m} - \mu) - \frac{\tilde{\phi}^2}{4} \frac{k_{\text{F}}^2}{m} \log(c/k_{\text{F}})] \tau} n_{\text{F}}(q) \Theta(-\tau) \\ &\quad + e^{[-(\frac{q^2}{2m} - \mu) + \frac{\tilde{\phi}^2}{4} \frac{k_{\text{F}}^2}{m} \log(c/k_{\text{F}})] \tau} (1 - n_{\text{F}}(q)) \Theta(\tau). \end{aligned} \quad (51)$$

Here  $n_{\text{F}}(q)$  is the Fermi factor  $n_{\text{F}}(q) = 1/(\exp[\beta(q^2/(2m) - \mu)] + 1)$ . When carrying out the Fourier transformation with respect to  $\mathbf{q}$  and comparing the result with the asymptotics (42, 43) and (48) of the exact Chern-Simons Green's function we obtain that the two asymptotics are in accordance for small  $\tau$  and  $\mathbf{r} = 0$ . Furthermore, we see that the prefactor of the  $\log(c/k_{\text{F}})$  term in  $G^{\text{HF}}(q, \omega)$  is equal to the exact Green's function.

Next, we calculate the Chern-Simons Green's function in RPA. For doing this we use the path integral in [17] which includes the bosonic Chern-Simons fields. This path integral correspondence to the path integral of HLR [3] up to one additive term in the action. This additive term is necessary to reproduce the correct ordering of the operators [17] in the Chern-Simons Hamiltonian (2). In [17], we calculated the grand canonical potential  $\Omega_{\text{RPA}}$  from this path integral in RPA. In the following, we will calculate the RPA self energy through  $\Sigma^{\text{RPA}} = \delta\Omega_{\text{RPA}}/(\delta G)$ . After some calculation,  $\Sigma^{\text{RPA}}$  is given in the leading  $c$  order by

$$\begin{aligned} \Sigma^{\text{RPA}}(q, \omega) &= \\ &\frac{\tilde{\phi}^2 \mu}{2} \log \left( \frac{c}{k_{\text{F}}} \right) \frac{\left( \mathrm{i}\omega - \frac{q^2}{2m} + \mu \right)}{\left( \omega_c + \text{sgn} \left[ \frac{q^2}{2m} - \mu \right] \left( \frac{q^2}{2m} - \mathrm{i}\omega - \mu \right) \right)}. \end{aligned} \quad (52)$$

Here,  $\omega_c$  is given by  $B/m$ .  $\text{sgn}[\cdot]$  is the sign of the argument. We see from this self energy formula that the prefactor of the  $\log(c)$  term gets a non trivial frequency dependence. We should mention that the asymptotic behaviour (52) of the RPA self energy is also correct in the case of taking into account the Coulomb interaction between the electrons.

As in the case of the Hartree-Fock self energy, we will calculate in the following the Fourier time transformation of  $\Sigma^{\text{RPA}}$ . For doing this, we have to solve the equation  $\mathrm{i}\omega - \frac{q^2}{2m} + \mu - \Sigma^{\text{RPA}} = 0$ , representing a quadratic equation in the frequencies. The two frequency solutions correspond to two additive terms of the Fourier transform of  $\Sigma^{\text{RPA}}$ . One of the solutions of this quadratic equation is given by  $\mathrm{i}\omega = (2n_{\text{F}}(q) - 1) \tilde{\phi}^2 \mu \log(c/k_{\text{F}})$ . This solution corresponds to a term in the Fourier transformed RPA Green's function which has the correct asymptotics of the exact Green's function for  $c \rightarrow 0$ . The other solution is finite for  $c \rightarrow 0$ . Thus, the corresponding additive term in the RPA Green's function is finite for  $c \rightarrow 0$  (modulo logarithmic singularities). Summarizing, the RPA Green's function has not the asymptotic behaviour of the exact Chern-Simons Green's function for  $A \rightarrow \infty$ .

So far, we did not take into account the self consistence of the approximation of the Green's function. Usually one has to use a self consistent approximation of the Green's function in a perturbation theory (*e.g.* [18]). Thus, we will calculate in the following the self consistent Hartree-Fock Green's function in the leading  $c$  order. The self consistent Hartree-Fock self energy in the leading  $c$  order is a solution of the following equation

$$\Sigma_{sc}^{\text{HF}}(q) = \frac{\tilde{\phi}^2 k_F^2}{4m} \left[ \log\left(\frac{c}{k_F}\right) n_F(q, \Sigma_{sc}^{\text{HF}}(q)) - \log\left(\frac{c}{k_F}\right) (1 - n_F(q, \Sigma_{sc}^{\text{HF}}(q))) \right] + \Sigma_{sc}^f(q). \quad (53)$$

Here  $n_F(q, \Sigma_{sc}^{\text{HF}})$  is the Fermi factor  $n_F(q, \Sigma_{sc}^{\text{HF}}) = 1/(\exp[\beta(q^2/(2m) + \Sigma_{sc}^{\text{HF}}(q) - \mu)] + 1)$ .  $\Sigma_{sc}^f(q)$  is for  $c \rightarrow 0$  the finite part of the Hartree-Fock self energy calculated with the self consistent Hartree-Fock Green's function  $G_{sc}^{\text{HF}}(\mathbf{q}, \omega) = -1/(i\omega - q^2/(2m) - \Sigma_{sc}^{\text{HF}}(q) + \mu)$ . In the leading  $c$  order this self consistent equation is solved by ( $T = 0$ )

$$\Sigma_{sc}^{\text{HF}}(q, k_F^*) = \frac{\tilde{\phi}^2 (k_F^*)^2}{4m} \log\left(\frac{c}{k_F^*}\right) (2\Theta(q - k_F) - 1) \quad (54)$$

provided that the Fermi momentum  $k_F^*$  solves the following equation

$$\frac{(k_F^*)^2}{2m} - \mu + \Sigma^f(k_F^*, k_F^*) = 0. \quad (55)$$

In order to fix  $k_F^*$ , we do not have to calculate the finite part  $\Sigma^f(q, k_F^*)$  of the Hartree-Fock self energy. By using the equation  $\lim_{\beta \rightarrow \infty} \beta \int d^2k n_F(k) (1 - n_F(k)) F(k) = \int d^2k \delta(|k| - k_F) F(k)$  we get that  $\Sigma^f(k_F^*, k_F^*)$  is given by

$$\Sigma^f(k_F^*, k_F^*) = \frac{(2\pi)}{m} \frac{\partial}{\partial \mu} U^{\text{HF}}((k_F^*)^2/(2m), B) - \left(\frac{(k_F^*)^2}{2m}\right). \quad (56)$$

$U^{\text{HF}}(\mu, B)$  is the Hartree-Fock energy of the Chern-Simons Hamiltonian (2) (containing the kinetic energy). We employed in (56) the mathematic notation for the ordering of the derivation and insertion of the arguments of the functions. This means for equation (56) that we have to partially derivate at first the function  $U^{\text{HF}}(\mu, B)$  depending on the variables  $(\mu, B)$ . Afterwards we have to insert the expressions given in the function brackets.  $B$  is the external magnetic field. The Hartree-Fock energy of the  $\nu = 1/\tilde{\phi}$  system is given by [19,20]

$$U^{\text{HF}}(\mu, \tilde{\phi}m\mu) = \frac{m}{4\pi}\mu^2 + \frac{3m}{16\pi}\tilde{\phi}^2\mu^2. \quad (57)$$

Since we calculate  $U^{\text{HF}}(\mu, B)$  in (57) for  $B = \tilde{\phi}m\mu$  it is not correct to insert (57) in (56). This was shown in [21] generally in the case of the determination of  $\mu$  through the equation  $-\partial\Omega/(\partial\mu) = N/A$ . Here  $\Omega$  is the grand canonical potential of the Chern-Simons system. We obtained in

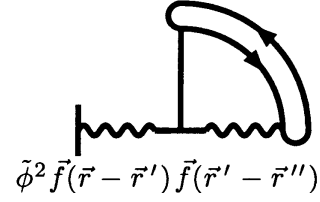


Fig. 2. The self energy graph  $\Sigma_c$ .

the paper [21] that one gets a correction to this equation if  $\Omega$  is calculated under the constraint  $B = 2\pi\tilde{\phi}N/A$ . With the help of the derivations in this paper it is easy to see that the following equation results in the correct  $k_F^*$

$$\Sigma^f(k_F^*, k_F^*) = \frac{(2\pi)}{m} \frac{\partial}{\partial \mu} U_{\nu=1/\tilde{\phi}}^{\text{HF}}\left(\frac{(k_F^*)^2}{2m}\right) + \Sigma_c(k_F^*) - \left(\frac{(k_F^*)^2}{2m}\right). \quad (58)$$

Here  $U_{\nu=1/\tilde{\phi}}^{\text{HF}}(\mu)$  is given by  $U_{\nu=1/\tilde{\phi}}^{\text{HF}}(\mu) := U^{\text{HF}}(\mu, \tilde{\phi}m\mu)$ .

The self energy diagram  $\Sigma_c$  is shown in Figure 2. By closing the open ends of the self energy diagram by an interaction free Green's function we see that the resulting  $\Omega$  diagram is canceled under the condition  $B = \tilde{\phi}m\mu$  with a diagram where this Green's function is replaced by a coupling to the external magnetic field  $B$ . Furthermore, we see from the figure that the momentum transfer to the external coupling is zero for an infinite system. Thus, we get difficulties in calculating this diagram (because  $\lim_{q \rightarrow 0} \mathbf{f}(q) = \infty$ ). We showed in [21] that the correct way to calculate this diagram is to calculate the integrals of the diagram for a finite system taking at last the limit  $A \rightarrow \infty$ . By doing this we get

$$\Sigma_c(q) = -\frac{1}{m} \frac{(2\pi\tilde{\phi})^2}{(2\pi)^4} \times \lim_{q \rightarrow 0} \int d^2k_1 d^2k_2 n_F(|\mathbf{k}_1 - \mathbf{k}_2|) n_F(|\mathbf{k}_2 - \mathbf{q}|) \frac{\mathbf{k}_1}{|\mathbf{k}_1|^2} \frac{\mathbf{q}}{|\mathbf{q}|^2}. \quad (59)$$

To calculate the double integral we expand  $n_F(|\mathbf{k}_2 - \mathbf{q}|)$  for small  $q$  and  $T > 0$ . By carrying out the integration we get

$$\Sigma_c(q) = -\tilde{\phi}^2\mu. \quad (60)$$

By inserting (57) and (60) in equation (58) we get with the help of (55)

$$\frac{(k_F^*)^2}{2m} = \frac{\mu}{(1 + \frac{\tilde{\phi}^2}{4})}. \quad (61)$$

In the following, we will make a similar calculation neglecting  $\Sigma_c(q)$  in (58). We denote  $\tilde{k}_F^*$  by the Fermi momentum which is calculated through the equations (55, 58) with  $\Sigma_c(k) = 0$ . This Fermi momentum is given by

$$\frac{(\tilde{k}_F^*)^2}{2m} = \frac{\mu}{(1 + \frac{3}{4}\tilde{\phi}^2)}. \quad (62)$$

Now we calculate the density of the system by using the self-consistent Hartree-Fock Green's function. With the help of  $G_{sc}^{HF}(\mathbf{q}, \omega, k_F^*) = -1/(i\omega - q^2/(2m) - \Sigma_{sc}^{HF}(q, k_F^*) + \mu)$  we get for the electron density  $N/A = \sum_{\omega} \sum_{\mathbf{q}} G_{sc}^{HF}(\mathbf{q}, \omega, k_F^*) e^{i\omega\eta}$  ( $\eta$  is an infinite small positive parameter) with  $\mu = (2\pi)mN/A$  for the  $\nu = 1/2$  system we get  $1/2$  of the electron density. In contrast to this the trace over the exact Green's function is given by the electron density. In the paper [21], we showed that  $-\partial\Omega'(\mu, \infty)/(\partial\mu) = 0$ .  $\Omega'(\mu, \beta)$  is the grand canonical potential of the  $\nu = 1/\tilde{\phi}$  system which is calculated under the constraint  $B = (2\pi\tilde{\phi})N/A$ . From this equation it is clear that  $\sum_{\omega} \sum_{\mathbf{q}} G_{sc}^{HF}(\mathbf{q}, \omega, \tilde{k}_F^*) e^{i\omega\eta}$  should be zero to be a good approximation. For the  $\nu = 1/2$  system, we get for this expression with  $\mu = (2\pi)mN/A$ ,  $1/4$  of the electron density. Thus, we get the correct relation between the densities calculated with the help of the Hartree-Fock Green's function  $G_{sc}^{HF}(\mathbf{q}, \omega, k_F^*)$  and  $G_{sc}^{HF}(\mathbf{q}, \omega, \tilde{k}_F^*)$ . When formulating a perturbation theory for the grand canonical potential which uses the Hartree-Fock Green's function as the mean field Green's function we have to use  $G_{sc}^{HF}(\mathbf{q}, \omega, \tilde{k}_F^*)$  because (as discussed above) insertions of the self energy  $\Sigma_c$  in  $\Omega$  diagrams are cancelled by  $B$  coupling diagrams. The formulation of this theory will be published in a later paper.

Until now we did not take into account the Coulomb interaction. By this interaction we obtain in the Hartree-Fock approximation an additional Fock diagram including the Coulomb vertex. This Coulomb diagram is finite. Thus we get also in this case an asymptotic Green's function of the form (51). This is also in agreement with the results of Section 2.5. At last, we have to mention that we obtain a difference for  $\Sigma_c$  from the result of Sitko and Jacak [12] (they calculated  $\Sigma_c = 0$ ). The reason is that they did not take the limit  $A \rightarrow \infty$  at the end of the calculation of  $\Sigma_c$ .

## 4 Conclusion

In this paper, we calculated the asymptotic form of the  $\nu = 1/\tilde{\phi}$  Chern-Simons Green's function for an infinite area  $A$  non-perturbationally. This was done concretely for the Coulomb free  $\nu = 1/\tilde{\phi}$  Chern-Simons theory. We obtain that the asymptotics of the Green's function behaves as  $G(\mathbf{r}, \tau) \sim e^{-f(r, \tau) \log(A)}$ . Due to the sign of  $\tau$  the function  $f(r, \tau)$  can be written as two different power expansions in  $\tau$ . We calculated  $f(r, \tau)$  to the third order in  $\tau$ . Due to this calculation we get that  $f(r, \tau)$  results in a positive function. It would be interesting to see if this is also true considering higher powers of  $\tau$ . Next, we discussed the asymptotic behaviour of the Green's function for the Chern-Simons theory taking into consideration the Coulomb interaction. We obtain (for temperature  $T > 0$ ) the same asymptotics as for the Green's function of the Chern-Simons theory without Coulomb interaction (in this case  $f(r, \tau)$  depends also on the Coulomb coupling constant  $e^2$ ).

In Section 3, we examined the Green's function of the Chern-Simons theory in the Hartree-Fock approximation

as well as in the RPA without Coulomb interaction. We obtained that the asymptotics of the Hartree-Fock approximation of the Green's function behaves for  $\mathbf{r} = 0$  and small  $\tau$  similar to the asymptotics of the exact one. Especially the prefactor of the  $\log(A)$  term of the Hartree-Fock Green's function is the same as of the exact Green's function. Next, we calculated the asymptotic behaviour of the RPA Green's function. We showed that this Green's function is finite for  $A \rightarrow \infty$ . This is not in correspondence with the exact Green's function. On the way to formulate a perturbation theory around the Hartree-Fock mean field, we examined the self consistent Hartree-Fock Green's function. We solved the self consistence equation in the leading  $1/A$  order. We obtain a self consistent Hartree-Fock Green's function which behaves similar to the Hartree-Fock Green's function with the difference that it has a different Fermi momentum. We obtained that the density of the electrons calculated with the help of the self consistent Hartree-Fock Green's function is  $1/2$  of the exact density (for the  $\nu = 1/2$  system). Furthermore, we calculated the self consistent Green's function without one of the Hartree-Fock self energy diagrams which is zero when inserted in  $\Omega$  diagrams. The electron density which is calculated with the help of this Green's function is  $1/4$  of the exact electron density (for the  $\nu = 1/2$  system). It was shown by us in [21] that the electron density calculated with this truncated Green's function should be zero. At last we obtained that the asymptotics of the Green's function in the Hartree-Fock approximation by taking into account the Coulomb interaction has the same form as the Green's function without Coulomb interaction. This is in accordance with the exact results of Section 2.

By taking the asymptotics of the Green's function seriously with respect to the principles of perturbational many-body theory we now have two options to go further. First, we may establish a theory which integrates the  $\log(A)$  singularity by using the Hartree-Fock Green's function as the mean field Green's function. This theory was formulated by us in [20]. It is our purpose to publish the results in a subsequent paper. Second, we may establish other formulations of the Chern-Simons theory (*i.e.* [9, 11]) in the hope to get a well behaved Green's function. One may think that this could be reached by the theory of Shankar and Murthy [9] speculated earlier. This assumption is in contrast to [10] where it was shown in RPA that one gets also a singular self energy in this theory (for temperature  $T = 0$ ).

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## Appendix A: The commutators for the calculation of the Chern-Simons Green's function

With the help of the equation (16) we get from (7) to order  $\tau^3$  cumulant expectation values of the operator  $H_{N+1} + \overline{H}_{N+1}$  of the form  $\sum_{u_p \in u_{0,k}} 1/p + O(1/p^2)$  and  $\sum_{u_p, u_{p+1} \in u_{0,k}} 1/p + O(1/p^2)$ . The terms of the form  $\sum_{u_p \in u_{0,k}} 1/p + O(1/p^2)$  are given by (we write down only these terms of the expectation value which scales as  $\sum 1/p$ ).

$$\begin{aligned}
\langle u_{0,\mathbf{k}} | [H_{4,1}, [H_{0,1}, H_{4,1}]] | u_{0,\mathbf{k}} \rangle_c &= \tilde{\phi}^2 \frac{1}{2} \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}, \\
\langle u_{0,\mathbf{k}} | [H_{0,1}, H_{4,1}] (H_{4,1} + H_3) | u_{0,\mathbf{k}} \rangle_c &= -\tilde{\phi}^2 \frac{1}{4} \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}, \\
\langle u_{0,\mathbf{k}} | (H_{4,1} + H_3) [H_{0,1}, H_{4,1}] | u_{0,\mathbf{k}} \rangle_c &= \tilde{\phi}^2 \frac{1}{4} \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}, \\
\langle u_{0,\mathbf{k}} | [H_{4,2}, [H_{0,1}, H_{4,1}]] | u_{0,\mathbf{k}} \rangle_c &= \tilde{\phi}^2 \frac{3}{8} \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}, \\
\langle u_{0,\mathbf{k}} | H_{4,2} [H_{0,1}, H_{4,1}] | u_{0,\mathbf{k}} \rangle_c &= \tilde{\phi}^2 \frac{3}{8} \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}, \\
\langle u_{0,\mathbf{k}} | [H_{4,1}, [H_{0,2}, H_{4,1}]] | u_{0,\mathbf{k}} \rangle_c &= \tilde{\phi}^2 \frac{1}{8} \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}, \\
\langle u_{0,\mathbf{k}} | [H_{0,2}, H_{4,1}] (H_{4,1} + H_3) | u_{0,\mathbf{k}} \rangle_c &= -\tilde{\phi}^2 \frac{1}{8} \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}, \\
\langle u_{0,\mathbf{k}} | [H_{4,1}, [H_{0,2}, H_{4,2}]] | u_{0,\mathbf{k}} \rangle_c &= \tilde{\phi}^2 \frac{1}{8} \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}, \\
\langle u_{0,\mathbf{k}} | [H_{0,2}, H_{4,2}] (H_{4,1} + H_3) | u_{0,\mathbf{k}} \rangle_c &= -\tilde{\phi}^2 \frac{1}{8} \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}, \\
\langle u_{0,\mathbf{k}} | [H_{4,2}, [H_{0,2}, H_{4,1}]] | u_{0,\mathbf{k}} \rangle_c &= \tilde{\phi}^2 \frac{1}{8} \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}, \\
\langle u_{0,\mathbf{k}} | H_{4,2} [H_{0,2}, H_{4,1}] | u_{0,\mathbf{k}} \rangle_c &= \tilde{\phi}^2 \frac{1}{8} \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}, \\
\langle u_{0,\mathbf{k}} | [H_{4,2}, [H_{0,2}, H_{4,2}]] | u_{0,\mathbf{k}} \rangle_c &= \tilde{\phi}^2 \frac{1}{8} \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}, \\
\langle u_{0,\mathbf{k}} | H_{4,2} [H_{0,2}, H_{4,2}] | u_{0,\mathbf{k}} \rangle_c &= \tilde{\phi}^2 \frac{1}{8} \sum_{u_p \in u_{0,\mathbf{k}}} \frac{1}{p}.
\end{aligned} \tag{A1}$$

We get the remaining terms of the form  $\sum_{u_p, u_{p+1} \in u_{0,k}} 1/p + O(1/p^2)$  of the cumulant expectation value of the operator  $H_{N+1} + \overline{H}_{N+1}$  by dropping the first three terms in the above list and substituting  $\sum_{u_p \in u_{0,k}}$  by  $-\sum_{u_p, u_{p+1} \in u_{0,k}}$ .

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